AD A052158

AD No.
DDC FILE COPY

18

TRASANA

9 19 TM-

TECHNICAL MEMORANDUM 2-77

6

THEORY OF STOCHASTIC DUELS – MISCELLANEOUS RESULTS

10 PREPARED BY
C. J. ANCKER, JR.
University of Southern California

11 MAR[redacted]78

12 44p.

15 DAHC04-72-A-0001

DEPARTMENT OF THE ARMY
US ARMY TRADOC SYSTEMS ANALYSIS ACTIVITY
WHITE SANDS MISSILE RANGE
NEW MEXICO 88002

361 550

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) TECHNICAL MEMORANDUM 2-77, "THEORY OF STOCHASTIC DUELS — MISCELLANEOUS RESULTS"		5. TYPE OF REPORT & PERIOD COVERED
		6. PERFORMING ORG. REPORT NUMBER USA TRASANA
7. AUTHOR(s) C. J. ANCKER, JR University of Southern California		8. CONTRACT OR GRANT NUMBER(s) DAHC04-72-A-0001; Task Order 76-10
9. PERFORMING ORGANIZATION NAME AND ADDRESS US Army TRADOC Systems Analysis Activity ATTN: ATAA-TG White Sands Missile Range, NM 88002		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS
11. CONTROLLING OFFICE NAME AND ADDRESS US Army Research Office Box CM, Duke Station Durham, NC 27706		12. REPORT DATE March 1978
		13. NUMBER OF PAGES 42
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) Battelle Columbus Laboratories P.O. Box 8832 Durham, NC 27707		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release - distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This memorandum presents particular applications of various aspects of the theory of stochastic duels that the author has accumulated over a period of years.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

DISCLAIMER

The findings in this memorandum are not to be construed as an official Department of the Army position.

DEPARTMENT OF THE ARMY
US ARMY TRADOC SYSTEMS ANALYSIS ACTIVITY
WHITE SANDS MISSILE RANGE
NEW MEXICO 88002

FOREWORD

The funds for supporting the preparation of this memorandum were provided by the US Army TRADOC Systems Analysis Activity through the Army Research Office, Short Term Scientific Assisiance Program. The program is administered by the Battele Columbus Laboratories, Inc.

TABLE OF CONTENTS

TRASANA MEMORANDUM 2-77

THEORY OF STOCHASTIC DUELS — MISCELLANEOUS RESULTS

1.0 PURPOSE

This memorandum documents mathematical derivations that have application in the use and further development of the theory of stochastic duels.

2.0 RESULTS

2.1 Introduction

Presented below are particular applications of various aspects of the theory of stochastic duels that the author has accumulated during the past 20 years. Some of the material dates back to his employment with what was formerly the US Army Operations Research Office that operated through the Johns Hopkins University and later with the System Development Corporation, Santa Monica, CA.

2.2 Marksman Problem with Erlang n Firing Time Distribution

a. The Gamma probability density function is expressed by

$$f(t) = \frac{t^{b-1} e^{-t/a}}{a^b \Gamma(b)}, \quad t \geq 0 \quad a, b > 0$$

$$= 0 \text{ , elsewhere}$$

If the parameter b is restricted to the set of positive integers $b = n$, $(n = 1, 2, \ldots)$, the function is normally called the Erlang distribution and is expressed by

$$f(t) = \frac{t^{n-1} e^{-t/a}}{a^n (n-1)!}, \quad t \geq 0 \; ; \; n = 1, 2, \ldots \quad a > 0$$

$$= 0 \quad \text{, elsewhere.}$$

b. The characteristic function of the Erlang n is

$$\phi(u) = \frac{1}{(1 - iau)^n} . \tag{1}$$

The characteristic function of the probability density function (PDF) of the time-to-hit for a Marksman firing at a passive target has been shown previously to be

$$\Phi(u) = \frac{p\phi(u)}{1 - q\phi(u)} \tag{2}$$

where p = marksman's fixed hit probability, and q = 1 - p.[1] Consequently, the PDF of the marksman's time-to-hit is

$$h(t) = \frac{p}{2\pi} \int_{-\infty}^{+\infty} \frac{\phi(u)e^{-itu}}{1 - q\phi(u)} \, du . \tag{3}$$

Substituting Equation (1) into Equation (3) results in

$$h(t) = \frac{p}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-itu}}{(1 - iau)^n - q} \, du . \tag{4}$$

The integrand of Equation (4) has n simple poles in the lower half of the complex plane which are

$$u_k = -\frac{i}{a}\left(1 - q^{1/n}\, e^{i2\pi k/n}\right) , \quad k = 0, 1, \ldots n-1 . \tag{5}$$

Using Equation (5), Equation (4) may be rewritten as

$$h(t) = \frac{p}{2\pi} \int_{-\infty}^{+\infty} \frac{(i/a)^n\, e^{-itu}}{\prod_{k=0}^{n-1} [u + i/a\,(1 - q^{1/n}\, e^{i2\pi k/n})]} \, du . \tag{6}$$

---

[1] *Trevor Williams and Clinton J. Ancker, Jr, Stochastic Duels, Operations Research, Volume XI, No. 5, October 1963, pp 803-817, Equation 14.*

This may be evaluated by integrating around the contour of Figure 1 by using residue theory since the integral on C vanishes as $R \to \infty$.

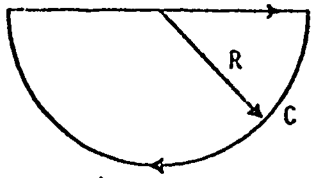


Figure 1 Contour of Integration

The jth residue is obtained by removing the jth singularity and replacing u by u_j. Finally, the residues are summed and multiplied by $-2\pi i$ since the contour is covered clockwise. This process provides the equation

$$h(t) = \frac{pe^{-t/a}}{aq^{\frac{n-1}{n}}} \sum_{j=0}^{n-1} \frac{e^{t/a\, q^{1/n} e^{i2\pi j/n}}}{\prod\limits_{\substack{k=o \\ k \neq j}}^{n-1} \left(e^{i2\pi j/n} - e^{i2\pi k/n}\right)} \tag{7}$$

where the product in the denominator is equal to 1 for n = 1.

c. If it is now noted that the mean of the PDF is 1/na and, following usual convention, that the rate of fire (r) is denoted by the reciprocal of the mean, Equation (7) becomes

$$h(t) = \frac{npre^{-nrt}}{q^{\frac{n-1}{n}}} \sum_{j=0}^{n-1} \frac{e^{nrtq^{1/n} e^{i2\pi j/n}}}{\prod\limits_{\substack{k=o \\ k \neq j}}^{n-1} \left(e^{i2\pi j/n} - e^{i2\pi k/n}\right)}, \quad n = 2,3\ldots \tag{8}$$

With more manipulation, Equation (8) may also be written

$$h(t) = \frac{npre^{-nrt}}{(2q^{1/n})^{n-1}} \sum_{j=0}^{n-1} \frac{(-1)^j e^{i2\pi j/n} e^{nrtq^{1/n} e^{i2\pi j/n}}}{\prod\limits_{\substack{k=o \\ k \neq j}}^{n-1} \sin \pi(k-j)/n}, \quad n = 2,3,\ldots \tag{9}$$

EXAMPLE 1: Letting n = 1 (remembering that the product in the denominator is 1 for n = 1), then

$$h(t) = p\, r\, e^{-prt} \tag{10}$$

and the distribution function (DF) is

$$H(t) = \int_0^t h\,(\xi)\, d\xi = 1 - e^{-prt}\,. \tag{11}$$

EXAMPLE 2: Letting n = 2

$$h(t) = \frac{2pre^{-2rt}}{\sqrt{q}}\left[\frac{e^{2r\sqrt{q}\,t} - e^{-2r\sqrt{q}\,t}}{2}\right] = \frac{2pre^{-2rt}}{\sqrt{q}} \sinh 2r\,\sqrt{q}\,t \tag{12}$$

and the DF by integration is

$$H(t) = 1 - \frac{e^{-2rt}}{\sqrt{q}}\left[\sinh 2r\,\sqrt{q}\,t + \sqrt{q}\cosh 2r\,\sqrt{q}\,t\right]. \tag{13}$$

d. In passing, it is also noted that the DF of the time-to-hit may be given in terms of characteristic functions as

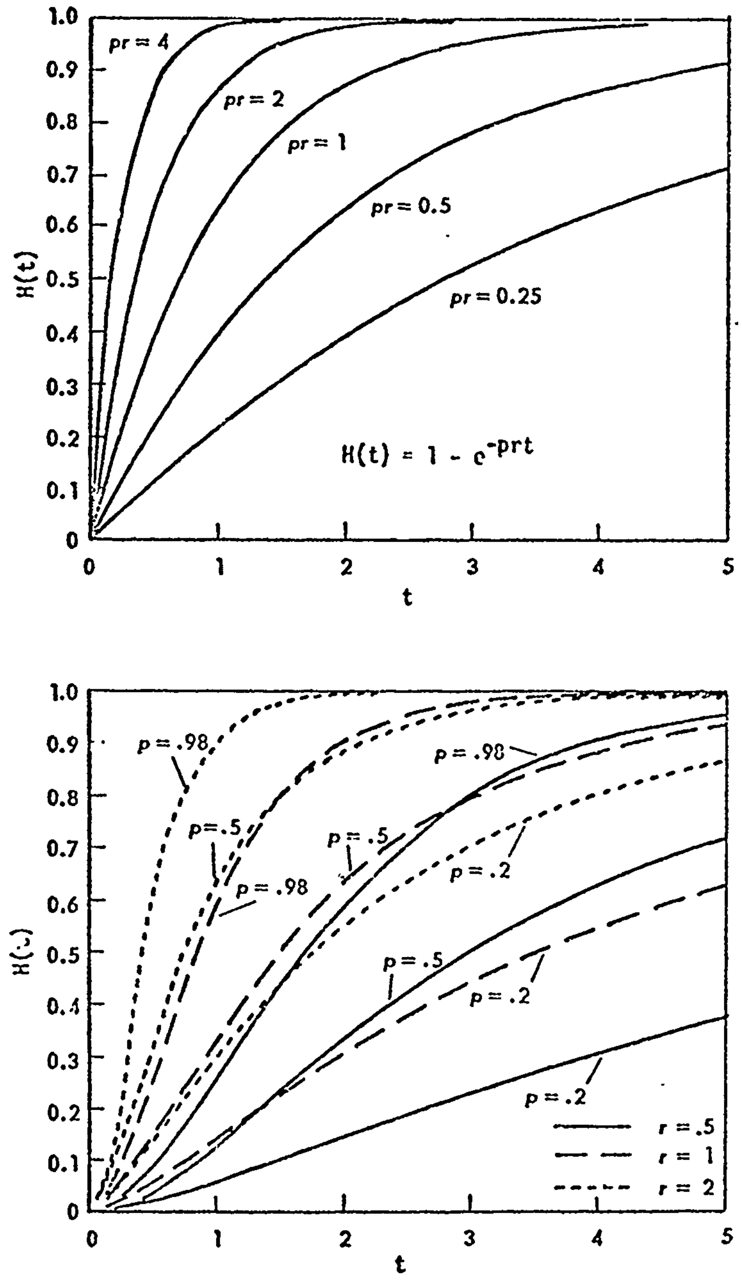
$$H(t) = \frac{p}{2\pi i}\int_{-\infty}^{+\infty} \frac{\phi\,(u)\,(1 - e^{-itu})}{[1 - q\,\phi(u)]\,u}\,du \tag{14}$$

by noting that $H(t) = \int_0^t h(\xi)\,d\xi$ and applying Equation (51) of para 2.7.3 a to this expression and to Equation (2). The plots of Equations (11) and (13) are shown in Figure 2 for comparison.

2.3 Tactical Equity Duel with Erlang 2 Firing Times[2]

a. For half of the time, A sights B first and fires a round that either kills or alerts B, in which case the duel proceeds as a fundamental

[2]*Ibid, p 811*



$H(t) = 1 - [\exp(-t?r)]/\sqrt{q}\ [\sin h^{2r\sqrt{q}\ t} + \sqrt{q} \cos h^{2r\sqrt{q}\ t}]$

Figure 2 Plots of Equations (11) and (13)

duel. The other half of the time starts with B firing first. Thus,[3]

$$P[A] = \frac{1}{2}\,[p_A + q_A\,P[A]_f] + \frac{1}{2}\,q_B\,P[A]_f \tag{15}$$

where the subscripts refer to the parameters of A and B, and $P[A]_f$ is the probability of A winning the fundamental duel.

b. If the known result of the fundamental duel with Erlang 2 firing times[4] is substituted into Equation (15), the expression becomes

$$P[A] = \frac{1}{2}\,p_A\left\{\frac{(p_A r_A^2 - p_B r_B^2)[2r_A^2 - (r_A^2 + r_B^2)p_B] + 4r_A r_B(r_A + r_B)[2r_A + (r_B - r_A)p_B]}{(p_A r_A^2 - p_B r_B^2)^2 + 4r_A r_B(r_A + r_B)(p_A r_A + p_B r_B)}\right\}\cdot \tag{16}$$

2.4 Different Tactical Equity Duel

a. The duel described in paragraph 2.3 above is modified for providing a different approach. As before, each contestant fires one round first half of the time. However, in this case, the opponent has a loaded weapon and immediately returns the fire with one round, thus precipitating the duel if both survive the opening engagement. From this description

$$\begin{aligned} P[A] &= \frac{1}{2}\left\{p_A + q_A q_B\,P[A]_f\right\} + \frac{1}{2}\left\{q_B p_A + q_B q_A\,P[A]_f\right\} \\ &= \frac{1}{2}\,p_A(1 + q_B) + q_A q_B\,P[A]_f \end{aligned} \tag{17}$$

where, as before, $P[A]_f$ is the probability that A wins the fundamental duel.

b. If the case of both sides having negative exponential firing times is substituted, the expression is

$$P[A] = \frac{1}{2}\,p_A(1 + q_B) + \frac{q_A q_B p_A r_A}{p_A r_A + p_B r_B}\ . \tag{18}$$

---

[3] *Ibid, Equation (28)*

[4] *Ibid, Equation (24)*

2.5 Duel with Random Initial Surprise and Laplace Sighting Times

a. The general solution of this duel is[5]

$$P[A] = \frac{1}{2\pi i} \int_L \Phi_A(-u)\ \Phi_B(u)\ \Theta(u)\ \frac{du}{u} \tag{19}$$

or

$$P[A] = 1 + \frac{1}{2\pi i} \int_U \Phi_A(-u)\ \Phi_B(u)\ \Theta(u)\ \frac{du}{u} \tag{20}$$

where $\int_L$ and $\int_U$ are the usual indented contours in the lower and upper half of the complex plane, and $\Phi_A(u)$, $\Phi_B(u)$, and $\Theta(u)$ are the characteristic functions of the time for A to hit, time for B to hit, and sighting time, respectively. A positive sighting time is an advantage for A, whereas a negative sighting time is an advantage for B.

b. If negative exponential firing times are assumed for A and B, and if the sighting time is allowed to be Laplace-distributed [g(t)], then

$$\Phi_A(-u) = \frac{p_A r_A}{p_A r_A + iu} \tag{21}$$

$$\Phi_B(u) = \frac{p_B r_B}{p_B r_B - iu} \tag{22}$$

$$g(t) = \frac{1}{2c}\, e^{-\frac{|t-d|}{c}} \qquad \begin{array}{c} -\infty < t < +\infty \\ c > 0 \\ -\infty < d < +\infty \end{array} \tag{23}$$

and

$$\Theta(u) = \frac{e^{idu}}{1 + c^2u^2} \tag{24}$$

[5]*Ibid, p. 813 et seq*

c. A typical PDF for the Laplace sighting time with d = 0 is shown in Figure 3. Note that the mean is d, and the standard deviation is $\sqrt{2}$ c.

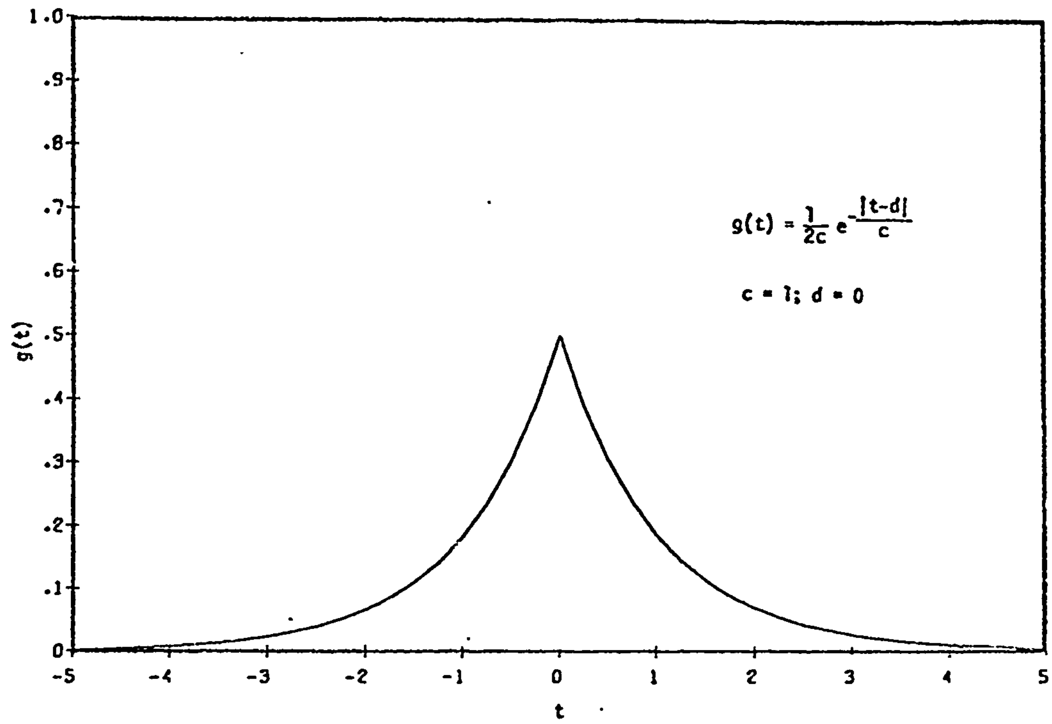


Figure 3 Laplace Sighting Time Probability Density Function

The two cases that must be distinguished are for positive d and negative d. For positive d, Equation (20) and the upper half of the complex plane are used. Thus,

$$P[A] = 1 + \frac{1}{2\pi i} \int_U \frac{p_A r_A \, p_B r_B \, e^{idu}}{(p_A r_A + iu)(p_B r_B - iu)(1 + c^2 u^2)} \frac{du}{u} \qquad (25)$$

There are 2 simple poles in the upper half plane, at $u = ip_A r_A$ and $u = i/c$. By applying the residue theorem, the result is

$$P[A] = 1 - \frac{p_B r_B \, e^{-p_A r_A d}}{(p_B r_B + p_A r_A)(1 - c^2 p_A^2 r_A^2)} - \frac{p_A r_A \, p_B r_B \, e^{-d/c}}{2(p_B r_B + \frac{1}{c})(p_A r_A - \frac{1}{c})} \qquad (26)$$

For negative d, Equation (19) and the lower half plane are used. Thus,

$$P[A] = \frac{1}{2\pi i} \int_L \frac{p_A r_A \, p_B r_B \, e^{idu}}{(p_A r_A + iu)(p_B r_B - iu)(1 + c^2 u^2)} \frac{du}{u} . \tag{27}$$

There are 2 simple poles in the lower half plane, at $u = -ip_B r_B$ and $u = -i/c$. By again applying the residue theorem, the result is

$$P[A] = \frac{p_A r_A \, e^{p_B r_B d}}{(p_A r_A + p_B r_B)(1 - c^2 p_B^2 r_B^2)} + \frac{p_A r_A \, p_B r_B \, e^{d/c}}{2(p_A r_A + \frac{1}{c})(p_B r_B - \frac{1}{c})} . \tag{28}$$

For the special case d = 0, Equation (26) or (28) is used for obtaining

$$P[A] = \frac{p_A r_A}{(\frac{1}{c} - p_B r_B)} \left[\frac{\frac{2}{c^2}(p_A r_A + \frac{1}{c}) - p_B r_B (p_A r_A + p_B r_B)(\frac{1}{c} + p_B r_B)}{2(p_A r_A + p_B r_B)(\frac{1}{c} + p_B r_B)(p_A r_A + \frac{1}{c})} \right] . \tag{29}$$

The plots of Equations (26), (28), and (29) for various values of the parameters $p_A r_A$, $p_B r_B$, c, and d are shown in Figure 4.

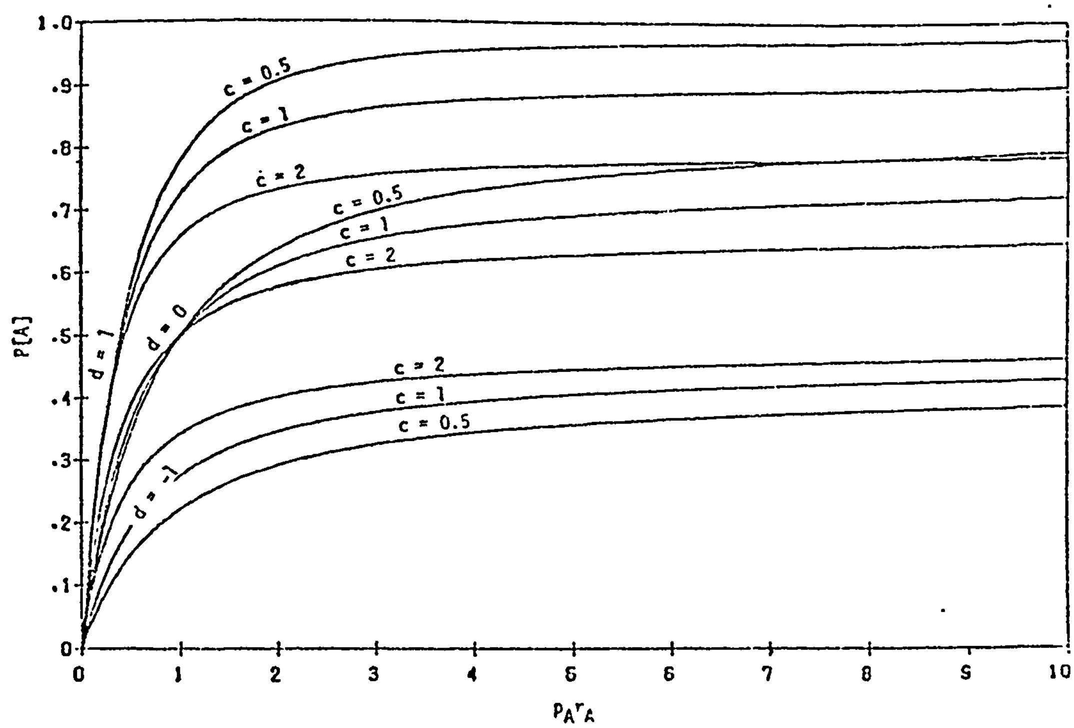


Figure 4 Stochastic Duels with Surprise — $p_B r_B = 1$

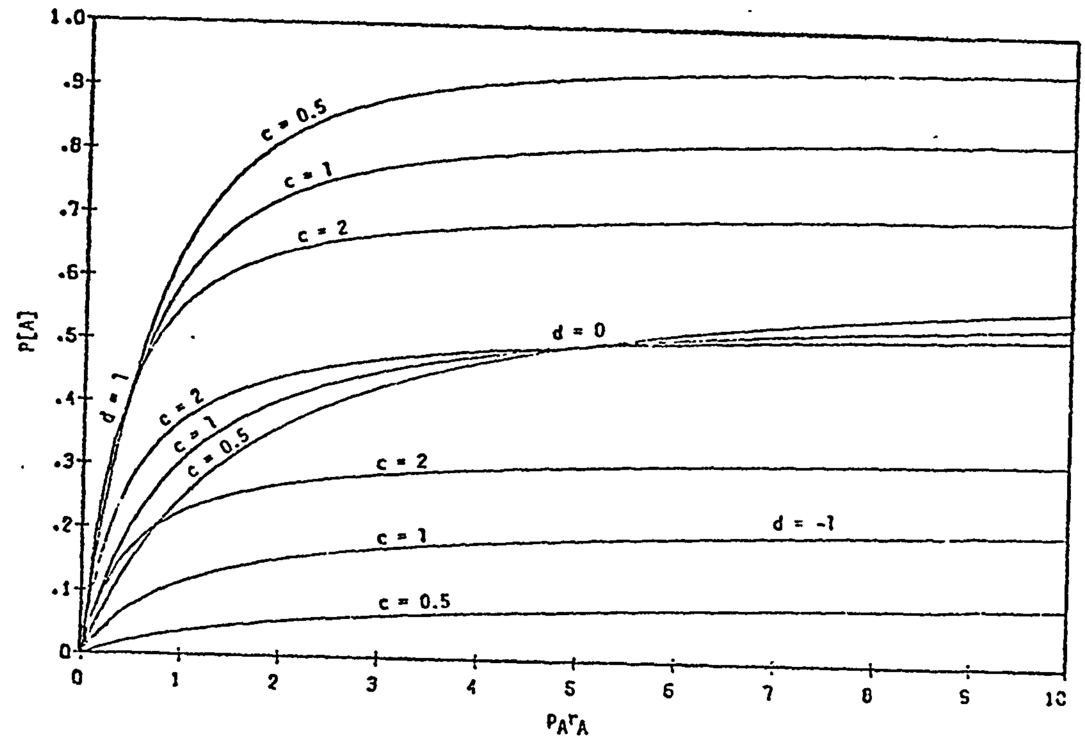


Figure 5 Stochastic Duels with Surprise — $p_B r_B = 5$

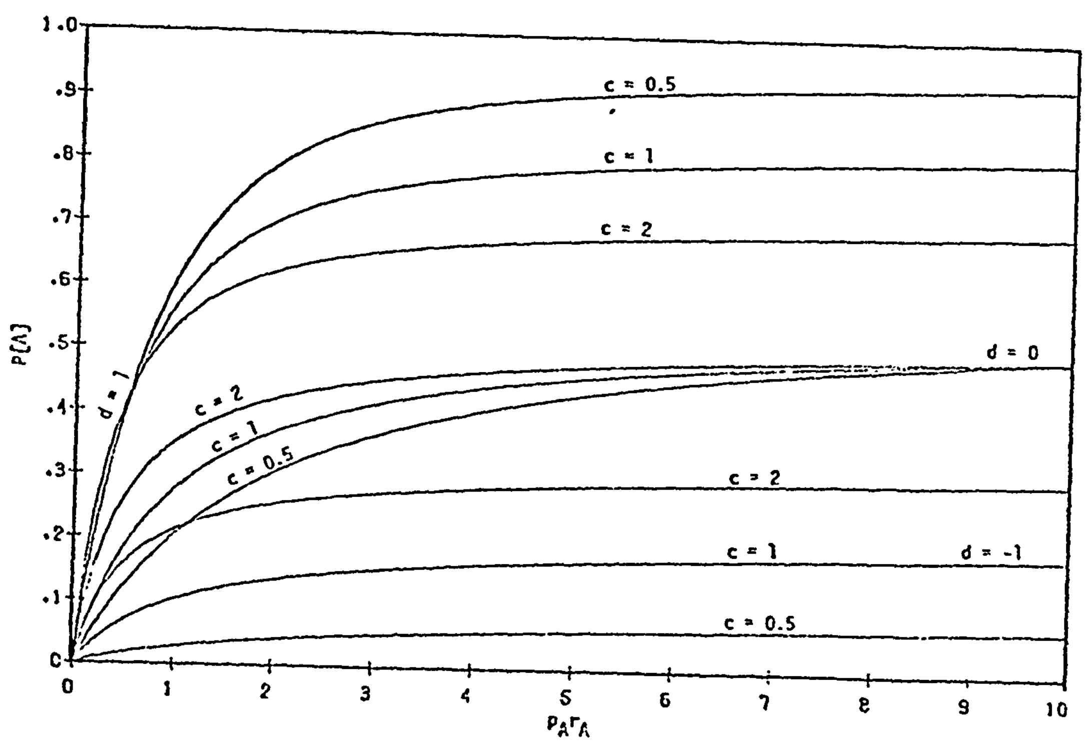


Figure 6 Stochastic Duels with Surprise — $p_B r_B = 10$

d. Two further simplifications may be noted. If it is supposed that A always has the sighting advantage and that $g(t) = 1/c\ e^{-t/c}$, $t > 0$, $c > 0$, and $g(t) = 0$ elsewhere, and $\Theta(u) = 1/(1-icu)$, it is easy to calculate as above and show that

$$P[A] = \frac{p_A r_A}{(p_A r_A + p_B r_B)(1 - p_B r_B c)} + \frac{p_A r_A\ p_B r_B}{(p_A r_A + \frac{1}{c})(p_B r_B - \frac{1}{c})} \ . \tag{30}$$

Similarly, if B always has the sighting advantage, with $g(t) = 1/c\ e^{t/c}$, $t < 0$, $c > 0$, and $g(t) = 0$ elsewhere, and $\Theta(u) = 1/(1 + icu)$; then

$$P[A] = \frac{p_A r_A}{(p_A r_A + p_B r_B)(1 + p_B r_B c)} \ . \tag{31}$$

In this case, obviously, P[A] has been reduced, compared to the fundamental duel, by a factor of

$$\frac{1}{1 + p_B r_B c}$$

2.6 Fundamental Duel with Erlang Firing Times

a. In this application, the fundamental duel is examined where A has an Erlang n firing time PDF, and B has an Erlang m PDF. This means that

$$\begin{aligned} f_A(t) &= \frac{t^{n-1}\ e^{-t/a}}{a^n\ (n-1)!} \ , \quad t > 0,\ a > 0 \\ & \qquad\qquad\qquad\quad n = 1, 2, \ldots \\ &= 0, \text{ elsewhere} \end{aligned} \tag{32}$$

with $\phi_A(u) = \dfrac{1}{(1 - iau)^n}$,

and

$$\begin{aligned} f_B(t) &= \frac{t^{m-1}\ e^{-t/b}}{b^m\ (m-1)!} \ , \quad t > 0,\ b > 0 \\ & \qquad\qquad\qquad\quad m = 1, 2, \ldots \\ &= 0, \text{ elsewhere} \end{aligned} \tag{33}$$

with $\phi_B(u) = \dfrac{1}{(1 - ibu)^m}$

From previous work, we have[6]

$$P[A] = \frac{1}{2\pi i} \int_{L} \frac{p_A}{[(1 + iau)^n - q_A]} \cdot \frac{p_B}{[(1 - ibu)^m - q_B]} \frac{du}{u} \tag{34}$$

which may be rewritten as

$$P[A] = \frac{p_A p_B}{2\pi i} \int_{L} \frac{1}{\left[(1 + iau)^n - q_A\right]} \cdot \frac{(\frac{i}{b})^m}{\left[(u + \frac{i}{b})^m - (\frac{i}{b})^m q_B\right]} \frac{du}{u} . \tag{35}$$

There are m simple poles in the lower half plane, at $u_k = -\frac{i}{b}(1 - q_B^{1/m} e^{i2\pi k/m})$, $k = 0, 1, \ldots, m-1$. Hence, Equation (35) may again be rewritten as

$$P[A] = \frac{p_A p_B}{2\pi i} \int_{L} \frac{1}{[(1 + iau)^n - q_A]} \cdot \frac{(\frac{i}{b})^m}{\prod_{k=0}^{m-1} [u + \frac{i}{b}(1 - q_B^{1/m} e^{i2\pi k/m})]} \frac{du}{u} . \tag{36}$$

b. The j^{th} singularity is now removed, and u_j is substituted for all u's to get the j^{th} residue. The residue theorem is then used, multiplying by $-2\pi i$ and summing over all residues. Finally, it is noted that $r_A = 1/na$ and $r_B = 1/mb$. Thus

$$P[A] = \frac{p_A p_B}{q_B^{(m-1)/m}} \sum_{j=0}^{m-1} \frac{1}{\left\{\left[1 + \frac{mr_B}{nr_A}\left(1 - q_B^{1/m} e^{i2\pi j/m}\right)\right]^n - q_A\right\} \left\{\prod_{\substack{k=0 \\ k \neq j}}^{m-1} \left(e^{i2\pi j/m} - e^{i2\pi k/m}\right)\right\} \left\{1 - q_B^{1/m} e^{i2\pi j/m}\right\}} \tag{37}$$

where it is noted that the product term in the denominator is 1 if $m = 1$. One can immediately check this result by substituting $m = n = 1$[7] and by letting $m = n = 2$.[8] Equation (37) above may be written in the following form which may be somewhat simpler to evaluate.

$$P[A] = \frac{p_A p_B}{(2q_B^{1/m})^{m-1}} \sum_{j=0}^{m-1} \frac{(-1)^j e^{i2\pi j/m}}{\left\{[1 + \frac{mr_B}{nr_A}(1 - q_B^{1/m} e^{i2\pi j/m})]^n - q_A\right\}(1 - q_B^{1/m} e^{i2\pi j/m}) \prod_{\substack{k=0 \\ k \neq j}}^{m-1} \sin \frac{\pi}{m}(k - j)} \tag{38}$$

and again the product term in the denominator is 1 if $m = 1$.

---

[6] *Ibid, Equations (14) and (20)*

[7] *Ibid, Equation (6)*

[8] *Ibid, Equation (24)*

2.7 Some Results in the Theory of Characteristic Functions (Modified Fourier Transforms) of Positive Random Variables

a. Shown below are results that may be useful in new applications of the Theory of Stochastic Duels. Only positive random variables are considered, i.e., PDFs such that

$$\left.\begin{aligned} f(t) &\geq 0, \quad t \geq 0 \\ &= 0, \quad t < 0 \\ \text{and} \qquad \int_0^\infty f(t)\,dt &= 1 \end{aligned}\right\} \tag{39}$$

with characteristic function

$$\left.\begin{aligned} \phi(u) &= \int_0^\infty e^{iut}\, f(t)\,dt \\ \text{and with inverse} \qquad f(t) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} e^{-iut}\,\phi(u)\,du \end{aligned}\right\}\ u \text{ real} \tag{40}$$

2.7.1 Parseval's Theorems

Three versions of Parseval's Theorem are denoted by

$$\int_{-\infty}^{+\infty} f_A(t)\, f_B(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi_A(-u)\,\phi_B(u)\,du \qquad (41)^9$$

$$\int_{-\infty}^{+\infty} e^{iut}\, f_A(t)\, f_B(t)\,dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \phi_A(u-w)\,\phi_B(w)\,dw \qquad (42)^{10}$$

---

9 *Titchmarsh, "Introduction to the Theory of Fourier Integrals," Oxford University Press, 2d Edition, 1948; Equation 2.1.1, p 50.*

10 *Ibid, Equation 2.1.8, p 51*

$$\int_{-\infty}^{+\infty} e^{iut} \; f_A(t)\; f_B(t)\; f_C(t)\; dt \;=\; \frac{1}{4\pi^2}\int_{-\infty}^{+\infty} \phi_A(u-w) \left(\int_{-\infty}^{+\infty} \phi_B(w-v)\; \phi_C(v)\; dv \right) dw \quad (43)$$

This result comes about by applying Equation (42) to the left-hand side twice.

N.B. These theorems do <u>*not*</u> *require the functions f(t) to be pdf's. They are general for any integrable functions.*

2.7.2 Properties of Characteristic Functions of Positive Random Variables

a. $$\phi(0) = 1 \quad (44)$$

PROOF: From Equations (39) and (40)

$$\phi(0) = \int_0^{\infty} e^{i(0)t} \; f(t)\; dt \;=\; \int_0^{\infty} f(t)\; dt \;=\; 1$$

b. $$|\phi(u)| \leq 1\,, \quad (\text{Imaginary } u \geq 0) \quad (45)$$

This implies no singularities in the upper half of the complex plane.

PROOF: Let $u = v + iw$, then

$$\phi(u) = \int_0^{\infty} e^{ivt}\; e^{-wt}\; f(t)\; dt$$

or

$$\left|\phi(u)\right| \leq \int_0^{\infty} e^{-wt}\; f(t)\; dt \quad , \quad \left|e^{ivt}\right| \leq 1$$

and

$$\left|\phi(u)\right| \leq \int_0^{\infty} f(t)\; dt = 1 \quad , \quad \left|e^{-wt}\right| \leq 1$$

for $w \geq 0$, i.e., positive imaginary u.

c. $|\phi(-u)| \leq 1$ (Imaginary $u \leq 0$) (46)

This implies no singularities in the lower half of the complex plane.

PROOF: Same as for subparagraph b above, with negative u where final step depends on $|e^{wt}| \leq 1$ for negative w.

d. $|\phi(u)| \leq \frac{k}{|u|}$ f(t) a differentiable function of bounded variation. Imaginary $u \geq 0$, k = positive constant. (47)

This implies that $\phi(u)$ diminishes as 1/R in the upper half-plane where R is the radius of a semicircular path of integration in the complex plane.

PROOF: Integrate Equation (40) by parts to obtain

$$\phi(u) = \frac{i\ f(0)}{u} + \frac{i}{u} \int_0^\infty e^{iut}\, f'(t)\, dt$$

This makes use of the fact that, necessarily, $f(\infty) = 0$. Now, for u complex and $t \geq 0$, $|e^{iut}| \leq 1$ in the upper half-plane. Thus,

$$|\phi(u)| \leq \frac{f(0)}{|u|} + \frac{1}{|u|} \int_0^\infty |f'(t)|\, dt$$

Now, assuming a differentiable function f(t) of bounded variation, then $\int_0^\infty |f'(t)|\, dt$ is bounded; therefore, $|\phi(u)| \leq \frac{k}{|u|}$ where k is a positive constant and imaginary u is positive.

e. $|\phi(-u)| \leq \frac{k}{|u|}$ f(t) a differentiable function of bounded variation. Imaginary $u \leq 0$, k = positive constant. (48)

This implies that $\phi(-u)$ diminishes as 1/R in the lower half-plane.

PROOF: Same as for subparagraph d above using negative u. In this case for u complex and $t \geq 0$, $|e^{-iut}| \leq 1$ in the lower half-plane.

f. $$|\phi(u-w)| \leq 1 \qquad \begin{matrix}(\text{Imaginary } u \geq 0)\\(\text{Imaginary } w \leq 0)\end{matrix} \tag{49}$$

This implies no singularities in the upper half of the complex u plane and the the lower half of the complex w plane.

PROOF: Let $u = \xi + i\eta$ and $w = x + iy$ so that

$$\phi(u-w) = \int_0^\infty e^{i(\xi-x)t}\, e^{(y-\eta)t}\, f(t)\, dt$$

and

$$|\phi(u-w)| \leq \int_0^\infty |e^{(y-\eta)t}|\, f(t)\, dt$$

$$\leq \int_0^\infty f(t) = 1$$

if $y-\eta \leq 0$ for all y,η. This implies y is negative and η is positive or that Imaginary u positive and Imaginary w is negative.

g. $$|\phi(u-w)| \leq \frac{k}{|u-w|} \tag{50}$$

f(t) a differentiable function of bounded variation. Imaginary $u \geq 0$, Imaginary $w \leq 0$, and k = positive constant.

This implies that $\phi(u-w)$ diminishes as 1/R in the upper half of the u plane and in the lower half of the w plane.

PROOF: Integrating the integral form of $\phi(u-w)$ by parts, again using $f(\infty) = 0$

$$\phi(u-w) = \frac{i\ f(0)}{u-w} + \frac{i}{u-w} \int_0^\infty e^{i(u-w)t} f'(t)\ dt$$

Now, $|e^{i(u-w)t}| \leq 1$ for Imaginary u positive and Imaginary w negative. Therefore,

$$|\phi(u-w)| \leq \frac{f(0)}{|u-w|} + \frac{1}{|u-w|} \int_0^\infty |f'(t)|\ dt$$

and again, if f(t) is a differentiable function of bounded variation

$$|\phi(u-w)| \leq \frac{k}{|u-w|} \quad \text{where } k = \text{positive constant.}$$

2.7.3 Additional Theorems

a.
$$\int_0^a e^{iut} f(t)\ dt = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\phi(w+u)(1 - e^{-iwa})}{w}\ dw \tag{51}$$

Note that for $u = 0$, an expression is also given for the distribution function of a random variable in terms of characteristic functions.

PROOF: Using Equation (40)

$$\int_0^a e^{iut} f(t)\ dt = \int_0^a e^{iut} \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iwt}\ \phi(w)\ dw \right) dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} \phi(w)\ dw \int_0^a e^{i(u-w)t}\ dt$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\phi(w)(1 - e^{i(u-w)a})}{w - u}\ dw$$

The desired result is now obtained by replacing ω with $(\omega + u)$.

b. $$\int_a^\infty e^{iut} f(t)\, dt = \phi(u) - \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\phi(\omega + u)(1 - e^{-i\omega a})}{\omega}\, d\omega \tag{52}$$

Note that for $u = 0$, this provides an expression for the complementary distribution function in terms of characteristic functions.

PROOF:

$$\int_a^\infty e^{iut} f(t)\, dt = \int_0^\infty e^{iut} f(t)\, dt - \int_0^a e^{iut} f(t)\, dt$$

$$= \phi(u) - \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\phi(\omega + u)(1 - e^{-i\omega a})}{\omega}\, d\omega$$

where we have used Equations (40) and (51).

c. $$\int_0^\infty e^{iut} \left(\int_t^\infty f(\xi)\, d\xi \right) dt = \frac{\phi(u) - 1}{iu} \tag{53}$$

This is the characteristic function of the complementary distribution function.

PROOF: Integrating by parts results in

$$\int_0^\infty e^{iut} \left(\int_t^\infty f(\xi)\, d\xi \right) dt = -\frac{1}{iu} + \frac{\phi(u)}{iu} = \frac{\phi(u) - 1}{iu}$$

d.

$$\int_0^{\infty} f_A(t)\left[\int_t^{\infty} f_B(\xi)\, d\xi\right] dt = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\phi_A(-u)\,[\phi_B(u) - 1]}{u}\, du \qquad (54)$$

PROOF: Previously given.[11]

e.

$$\int_0^{\infty} f_A(t)\left[\int_t^{\infty} f_B(\xi)\, d\xi\right]\left[\int_t^{\infty} f_C(\eta)\, d\eta\right] dt$$

$$= \frac{1}{4\pi^2}\int_{-\infty}^{+\infty} \frac{[\phi_C(-u) - 1]}{u}\left(\int_{-\infty}^{+\infty} \frac{\phi_A(u - w)[\phi_B(w) - 1]}{w}\, dw\right) du \qquad (55)$$

PROOF: Previously given.[12]

f.

$$\int_0^{\tau} f_A(t)\left(\int_t^{\infty} f_B(\xi)\, d\xi\right) dt = \frac{1}{4\pi^2}\int_{-\infty}^{+\infty} \frac{(e^{-iu\tau} - 1)}{u}\left(\int_{-\infty}^{+\infty} \frac{\phi_A(u - w)[\phi_B(w) - 1]}{w}\, dw\right) du \qquad (56)$$

PROOF: Previously given.[13]

g.

$$\int_{\tau}^{\infty} f_A(t)\left(\int_t^{\infty} f_B(\xi)\, d\xi\right) dt =$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\phi_A(-u)[\phi_B(u) - 1]}{u}\, du - \frac{1}{4\pi^2}\int_{-\infty}^{+\infty} \frac{(e^{-iu\tau} - 1)}{u}\left(\int_{-\infty}^{+\infty} \frac{\phi_A(u - w)[\phi_B(w) - 1]}{w}\, dw\right) du \qquad (57)$$

---

[11] *Trevor Williams and Clinton J. Ancker, Jr, "Stochastic Duels," Operations Research, Volume 11, No. 5, October 1963, pp 803-817, Equation (18).*

[12] *C. J. Ancker, Jr, "Stochastic Duels of Limited Time Duration," CORS Journal, Volume 4, No. 2, July 1966, pp 69-81, Equation (5).*

[13] *Ibid, Equation (28)*

PROOF:

$$\int_{\tau}^{\infty} f_A(t)\left(\int_{t}^{\infty} f_B(\xi)d\xi\right)dt$$

$$= \int_{0}^{\infty} f_A(t)\left(\int_{t}^{\infty} f_B(\xi)d\xi\right)dt - \int_{0}^{\tau} f_A(t)\left(\int_{t}^{\infty} f_B(\xi)d\xi\right)dt$$

$$= \text{d(above)} - \text{f(above)}$$

h.

$$\int_{0}^{\infty} f_A(t)\left(\int_{t+a}^{\infty} f_B(\xi)\,d\xi\right)dt = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{e^{-iau}\,\phi_A(-u)[\phi_B(u)-1]}{u}\,du \quad (58)$$

PROOF: Letting $t = \eta - a$,

$$\int_{0}^{\infty} f_A(t)\left(\int_{t+a}^{\infty} f_B(\xi)\,d\xi\right)dt = \int_{a}^{\infty} f_A(\eta-a)\left(\int_{\eta}^{\infty} f_B(\xi)\,d\xi\right)d\eta \quad (59)$$

the inner integral is the inverse transform of Equation (53)

$$= \int_{a}^{\infty} f_A(\eta-a)\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{e^{-iu\eta}\,[\phi_B(u)-1]}{iu}\,du\right)d\eta$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{[\phi_B(u)-1]}{u}\left(\int_{a}^{\infty} f_A(\eta-a)\,e^{-iu\eta}\,d\eta\right)du$$

Now, let $\eta - a = \rho$, so the above

$$= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{[\phi_B(u) - 1]}{u} \left(\int_0^{\infty} f_A(\rho)\, e^{-iu(\rho+a)}\, d\rho \right) du$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{-iau} \phi_A(-u) \frac{[\phi_B(u) - 1]}{u} du \qquad \text{by Equation (40).}$$

i.

$$\int_0^{\infty} f_A(t) \left(\int_{at+b}^{\infty} f_B(\xi)\, d\xi \right) dt = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} e^{-iub}\, \phi_A(-au) \frac{[\phi_B(u) - 1]}{u} du. \quad (60)$$

PROOF: Let $t = \frac{\eta - b}{a}$, then

$$\int_0^{\infty} f_A(t) \left(\int_{at+b}^{\infty} f_B(\xi)\, d\xi \right) dt = \int_b^{\infty} f_A \left(\frac{\eta - b}{a} \right) \left(\int_{\eta}^{\infty} f_B(\xi)\, d\xi \right) \frac{d\eta}{a}$$

As in subparagraph h above, the inner integral may be replaced to provide

$$= \int_b^{\infty} f_A \left(\frac{\eta - b}{a} \right) \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-iu\eta} [\phi_B(u) - 1]}{iu} du \right) \frac{d\eta}{a}$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{[\phi_B(u) - 1]}{u} \left(\int_b^{\infty} f_A\left(\frac{\eta - b}{a}\right) e^{-iu\eta} \frac{d\eta}{a} \right) du$$

Now, let $\frac{n-b}{a} = \rho$ to provide

$$= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{[\phi_B(u) - 1]}{u} \left(\int_0^{\infty} f_A(\rho)\, e^{-iu(a\rho+b)}\, d\rho \right) du$$

and by Equation (40)

$$= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{-iub}\, \phi_A(-au)[\phi_B(u) - 1]}{u}\, du$$

j.
$$\int_0^{\tau} f_A(t) \left(\int_{at+b}^{\infty} f_B(\xi)\, d\xi \right) dt$$

$$= \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \frac{e^{-iu\tau} - 1}{u} \left(\int_{-\infty}^{+\infty} \frac{\phi_A(u - a\omega)\, e^{-ib\omega}[\phi_B(\omega) - 1]}{\omega}\, d\omega \right) du \qquad (61)$$

PROOF: Let $t = \frac{n-b}{a}$, then

$$\int_0^{\tau} f_A(t) \left(\int_{at+b}^{\infty} f_B(\xi)\, d\xi \right) dt = \int_b^{at+b} f_A\left(\frac{n-b}{a}\right) \left(\int_n^{\infty} f_B(\xi)\, d\xi \right) \frac{dn}{a}$$

and replacing the inner integral

$$= \int_b^{at+b} f_A\left(\frac{n-b}{a}\right) \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-i\omega n}\, [\phi_B(\omega) - 1]}{i\omega}\, d\omega \right) \frac{dn}{a}$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{[\phi_B(\omega) - 1]}{\omega} \left(\int_b^{at+b} f_A\left(\frac{n-b}{a}\right) e^{-i\omega n} \frac{dn}{a} \right) d\omega$$

Now, let $\frac{\eta-b}{a} = \rho$

$$= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{[\phi_B(\omega) - 1]}{\omega} \left(\int_0^{\tau} f_A(\rho)\ e^{-i\omega(a\rho+b)} d\rho \right) d\omega$$

and using Equation (51)

$$= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{[\phi_B(\omega) - 1]}{\omega}\ e^{-i\omega b} \left(\frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\phi_A(u - a\omega)[1 - e^{-iu\tau}]}{u}\ du \right) d\omega$$

$$= \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \frac{[\phi_B(\omega) - 1]}{\omega}\ e^{-i\omega b} \left(\int_{-\infty}^{+\infty} \frac{[e^{-iu\tau} - 1]\ \phi_A(u - a\omega)}{u}\ du \right) d\omega$$

which provides the form of Equation (61) if the order of integration is reversed.

k.
$$\int_{\tau}^{\infty} f_A(t) \left(\int_{at+b}^{\infty} f_B(\xi)\ d\xi \right) dt =$$

$$= \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{e^{-iub}\ \phi_A(-au)[\phi_B(u) - 1]}{u}\ du$$

$$- \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \frac{[e^{-iu\tau} - 1]}{u} \left(\int_{-\infty}^{+\infty} \frac{\phi_A(u - a\omega)\ e^{-i\omega b}[\phi_B(\omega) - 1]}{\omega}\ d\omega \right) du \qquad (62)$$

PROOF: The integral may be written as

$$\int_0^{\infty} f_A(t) \left(\int_{at+b}^{\infty} f_B(\xi)\ d\xi \right) dt - \int_0^{\tau} f_A(t) \left(\int_{at+b}^{\infty} f_B(\xi)\ d\xi \right) dt$$

= Equation (60) - Equation (61)

$$\text{l. } \int_0^\infty e^{iut} f_A(t) \left(\int_t^\infty f_B(\xi)\, d\xi \right) dt = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\phi_A(u-w)[\phi_B(w) - 1]}{w}\, dw \tag{63}$$

PROOF: First apply Euation (42) and then Equation (53) to the left-hand side of Equation (63) to obtain the right-hand side immediately.

$$\text{m. } \int_0^\infty f_A(t) \left(\int_t^\infty f_B(\xi)\, d\xi \right) \left(\int_t^\infty f_C(\eta) \left[\int_\eta^\infty f_D(\rho)\, d\rho \right] d\eta \right) dt$$

$$= \frac{1}{8\pi^3 i} \int_{-\infty}^{+\infty} \frac{1}{u} \left(\int_{-\infty}^{+\infty} \frac{\phi_C(-u-v)[\phi_D(v) - 1]\, dv}{v} - 1 \right) \left(\int_{-\infty}^{+\infty} \frac{\phi_A(u-w)[\phi_B(w) - 1]\, dw}{w} \right) du \tag{64}$$

PROOF: First apply Equation (55) and then Equation (63) to the left-hand side of Equation (64); with slight rearrangement, one obtains the right-hand side immediately.

$$\text{n. } \int_0^\infty f_A(t) \left(\int_t^\infty f_B(\xi)\, d\xi \right) \left(\int_t^\infty f_C(\eta)\, d\eta \right) \left(\int_t^\infty f_D(\rho)\, d\rho \right) dt$$

$$= -\frac{1}{8\pi^3 i} \int_{-\infty}^{+\infty} \frac{[\phi_D(v) - 1]}{v} \left\{ \int_{-\infty}^{+\infty} \frac{[\phi_C(w-v) - 1]}{(w-v)} \left(\int_{-\infty}^{+\infty} \frac{\phi_A(-u)[\phi_B(u-w) - 1]}{(u-w)}\, du \right) dw \right\} dv \tag{65}$$

PROOF: Apply in succession Equations (42), (43), and (53) to the left-hand side of Equation (65), and with some rearrangement of terms, the right-hand side is readily obtained.

2.7.4 Two Theorems Useful in Numerical Integration

a. The fundamental duel has a solution (see footnote 6) provided by

$$P[A] = \frac{1}{2\pi i} \int_L \frac{\phi_A(-u)\phi_B(u)}{u} \, du \, . \tag{66}$$

The indicated contour implied by $\int_L$ can just as well look like Figure 7 as long as the indenture contains only the singularity at the origin.

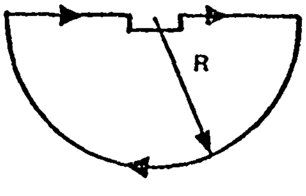


Figure 7 Lower Integration Contour

b. The investigation of certain properties of the integrand of Equation (66) is now desired. First, however, it is noted that $\phi_A(-u)\phi_B(u)$ is the characteristic function of $T_A - T_B$ where T_A and T_B are the random variables time to a hit for A and B, respectively. The difference is a random variable with range $(-\infty, +\infty)$ whose characteristic function is $\phi(u)$ with pdf f(t). Equation (66) may now be rewritten as

$$P[A] = \frac{1}{2\pi} \int_L \frac{\phi(u)}{iu} \, du = \frac{1}{2\pi} \int_L \frac{du}{iu} \int_{-\infty}^{+\infty} e^{iut} f(t) \, dt$$

$$= \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(t) \, dt \int_L \frac{e^{iut}}{iu} \, du \tag{67}$$

The substitution now made is $u = x + iy$ which is an analytic continuation of the real variable u into the complex plane. This substitution provides

$$P[A] = \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)\,dt \int_{L} \frac{e^{ixt}\,e^{-yt}}{x + iy}(dy - idx)$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} f(t)\,dt \left\{ \int_{L} \frac{e^{-yt}\left[(x \sin xt - y \cos xt)dx + (x \cos xt + y \sin xt)dy\right]}{x^2 + y^2} \right.$$

$$\left. -i \int_{L} \frac{e^{-yt}\left[(x \cos xt + y \sin xt)dx + (y \cos xt - x \sin xt)dy\right]}{x^2 + y^2} \right\} \tag{68}$$

c. Consider the two integrals within the brackets in Equation (68); specifically:

(1) By integrating along any line parallel to the x-axis at some y, e.g., y_o (which may be +, -, or zero, thus including the x-axis), $dy = 0$, and the integrand of the real part is even and the integrand of the imaginary part is odd.

(2) By integrating along lines parallel to the y-axis at some fixed x, e.g., x_o, $dx = 0$, and the integrands are neither even or odd. However, if two lines are considered, one at $+x_o$ and one at $-x_o$, it is noted that the integrand of the real part along $-x_o$ is the negative of the integrand of the real part along $+x_o$, i.e., the integrand is odd relative to the x variable, and the integrand of the imaginary part is exactly the same at $+x_o$ and $-x_o$, i.e., the integrand is even relative to the x variable.

d. The result of the above is that P[A] may be evaluated along the contour shown in Figure 8. The property described in c(1) above means that the imaginary part along C_1 is equal to the imaginary part along C_6 except that the signs are opposite; consequently, the sum is zero while the real parts are equal. The same applies for the integrals along C_3 and C_4.

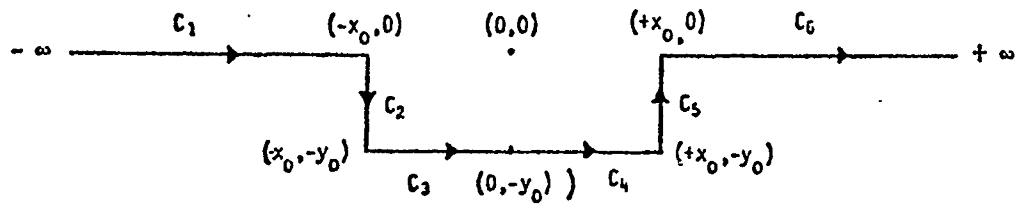


Figure 8 Numerical Integration Contour

e. From the property described in c(2) above, the imaginary part of C_2 is again canceled by the imaginary part of C_5, and the real parts are equal. Thus, the only thing that must be done is to integrate the real part of the integrand along C_4, C_5, and C_6 and multiply the sum by 2 to get P[A]. This can significantly reduce the amount of numerical integration.

f. One should also note that exactly the same argument holds for Equation (19) or for any integrand that is a product of characteristic functions with at least one positive argument and at least one negative argument divided by u. In addition, one may speculate that $-y_o$ and $+x_o$ should be chosen so as not to be too close to the singularity at the origin or to the nearest singularity in the lower half-plane. This would avoid steeply varying integrands and should reduce numerical integration difficulties. Also, one may simply continue C_4 to ∞, thus eliminating C_5 and C_6.

2.8 Generalizations of Some Results by Thompson

Thompson has derived some results for incorporating reliability into the theory of stochastic duels.[14] However, he has not simplified them by using characteristic functions. This has the disadvantages of leaving the solutions with one or more infinite sums of weighted convolutions of pdf's to be integrated. These operations can only be closed for very special pdf's and thus are of limited utility. By using characteristic functions, these sums can always be closed to a simple form, and the number of multiple integrations to be performed is reduced by at least one. Finally, there exists a characteristic function for every pdf and if it is unknown, can

[14] *David Thompson, Development of Analytical Models of Battalion Task Force Activities, Systems Research Laboratory, Department of Industrial Engineering, University of Michigan, Ann Arbor, Michigan, Report Number SRL19757 FR70-1 (U), Part F, Chapter 1, Editors: Seth Bonder and Robert Farrell.*

be approximated from real data for practical applications. For these reasons, we now simplify some of Thompson's results as follows:

a. In the case where either side's weapons may fail to function after some period of time and no withdrawals are permitted, Thompson provides the following expression for the probability that side A wins:[15]

$$P[A] = \int_0^\infty h_A(t)\left(\int_t^\infty r_A(\xi)\, d\xi\right)\left[1 - \int_0^t h_B(x)\left(\int_x^\infty r_B(\eta)\, d\eta\right) dx\right] dt \tag{69}$$

where $h_A(t)$ and $h_B(t)$ are the usual pdf's of A's and B's times to a kill, and $r_A(t)$ and $r_B(t)$ are A's and B's pdf's of times to weapons failure. All are independent of each other. We now rewrite Equation (69) in a more convenient form for our purposes, i.e.,

$$P[A] = \int_0^\infty h_A(t)\left(\int_t^\infty r_A(\xi)\, d\xi\right)\left[1 - \int_0^\infty h_B(x)\left(\int_x^\infty r_B(\eta)\, d\eta\right) dx\right] dt$$

$$+ \int_0^\infty h_A(t)\left(\int_t^\infty r_A(\xi)\, d\xi\right)\left(\int_t^\infty h_B(x)\left[\int_x^\infty r_B(\eta)\, d\eta\right) dx\right] dt \tag{70}$$

We now apply Equations (54) twice to the first expression on the right-hand side of Equation (70) and Equation (64) to the second expression to obtain

$$P[A] = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\phi_A(-u)[\Psi_A(u) - 1]}{u}\left\{1 - \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\phi_B(-w)[\Psi_B(w) - 1]}{w}\, dw\right\}$$

$$+ \frac{1}{8\pi^3 i}\int_{-\infty}^{+\infty} \frac{1}{u}\left(\int_{-\infty}^{+\infty} \frac{\phi_B(-u-v)[\Psi_B(v) - 1]\, dv}{v} - 1\right)\left(\int_{-\infty}^{+\infty} \frac{\phi_A(u-w)[\Psi_A(w) - 1]}{w}\, dw\right) du \tag{71}$$

where $\phi(\cdot)$ is the characteristic function of $h(t)$ and $\Psi(\cdot)$ is that of $r(t)$. Since all of the pdf's are for positive random variables, we may now use

[15] *Ibid p582, Equation 2*

the properties of positive RVs given earlier to further simplify these results to

$$P[A] = \frac{1}{2\pi i}\int_L \frac{\Phi_A(-u)\ \Psi_A(u)}{u}\,du\left[1 - \frac{1}{2\pi i}\int_L \frac{\Phi_B(-w)\ \Psi_B(w)}{w}\,dw\right]$$

$$+\ \frac{1}{8\pi^3 i}\int_U \frac{1}{u}\left(\int_L \frac{\Phi_B(-u-v)\ \Psi_B(v)}{v}\,dv\right)\left(\int_L \frac{\Phi_A(u-w)\ \Psi_A(w)}{w}\,dw\right)du\,. \quad (72)$$

The corresponding expression for a draw, P[AB], is provided by Thompson[16] as

$$P[AB] = \int_0^\infty r_A(t)\left[\int_t^\infty h_A(x)\ dx\right]dt \cdot \int_0^\infty r_B(t)\left[\int_t^\infty h_B(x)\ dx\right]dt. \quad (73)$$

Applying Equation (54) to each half of this product, we have

$$P[AB] = -\frac{1}{4\pi^2}\int_{-\infty}^{+\infty} \frac{\Psi_A(-u)[\Phi_A(u) - 1]}{u}\,du \cdot \int_{-\infty}^{+\infty}\frac{\Psi_B(-w)[\Phi_B(w) - 1]}{w}\,dw \quad (74)$$

which is now further simplified using the properties of positive random variables to

$$P[AB] = -\frac{1}{4\pi^2}\int_L \frac{\Psi_A(-u)\ \Phi_A(u)}{u}\,du \cdot \int_L \frac{\Psi_B(-w)\ \Phi_B(w)}{w}\,dw\,. \quad (75)$$

Thompson provides an example[17] where all the pdf's are negative exponentials. When these pdf's are used in Equations (72) and (75), we check out with his results except for an error in his expression for P[A]. He should have added a λ_A in the last parenthesis of the denominator.

---

[16] *Ibid, p582, Equation 3*

[17] *Ibid, p583*

b. In a situation where a withdrawal, and consequent end of the duel in a draw, occurs as soon as a weapon fails, Thompson provides[18]

$$\int_0^{\infty} h_A(t)\left(\int_t^{\infty} h_B(x)\ dx\right)\left(\int_t^{\infty} r_B(y)\ dy\right)\left(\int_t^{\infty} r_A(z)\ dz\right) dt \tag{76}$$

From Equation (65), we immediately have

$$P[A] = -\frac{1}{8\pi^3 i}\int_{-\infty}^{+\infty}\frac{[\Psi_A(v) - 1]}{v}\left[\int_{-\infty}^{+\infty}\frac{[\Psi_B(w-v) - 1]}{w-v}\left(\int_{-\infty}^{+\infty}\frac{\Phi_A(-u)[\Phi_B(u-w) - 1]}{u-w}\ du\right) dw\right] dv \tag{77}$$

which again may be reduced by the properties of positive RVs to

$$P[A] = -\frac{1}{8\pi^3 i}\int_L \frac{\Psi_A(v)}{v}\left[\int_L \frac{\Psi_B(w-v)}{w-v}\left(\int_L \frac{\Phi_A(-u)\Phi_B(u-w)}{u-w}\ du\right) dw\right] dv . \tag{78}$$

In Equation (78), the u integration is indented around $u = w$ on the real u-line, and the w integration is indented around $w = v$ on the real w-line. The results[19] for a draw, P[AB], are

$$P[AB] = \int_0^{\infty} r_A(t)\left(\int_t^{\infty} r_B(x)dx\right)\left(\int_t^{\infty} h_A(y)dy\right)\left(\int_t^{\infty} h_B(z)dz\right)$$

$$+\int_0^{\infty} r_B(t)\left(\int_t^{\infty} r_A(x)dx\right)\left(\int_t^{\infty} h_A(y)dy\right)\left(\int_t^{\infty} h_B(z)dz\right) . \tag{79}$$

[18] *Ibid, p584, Equation (4)*

[19] *Ibid, p585*

Again, we apply Equation (65) to each term of the right-hand side of Equation (79); thus

$$P[AB] = -\frac{1}{8\pi^3 i}\int_{-\infty}^{+\infty}\frac{[\phi_B(v)-1]}{v}\left[\int_{-\infty}^{+\infty}\frac{[\phi_A(w-v)-1]}{w-v}\left(\int_{-\infty}^{+\infty}\frac{\Psi_A(-u)[\Psi_B(u-w)-1]}{u-w}\,du\right)dw\right]dv$$

$$-\frac{1}{8\pi^3 i}\int_{-\infty}^{+\infty}\frac{[\phi_B(v)-1]}{v}\left[\int_{-\infty}^{+\infty}\frac{[\phi_A(w-v)-1]}{w-v}\left(\int_{-\infty}^{+\infty}\frac{\Psi_B(-u)[\Psi_A(u-w)-1]}{u-w}\,du\right)dw\right]dv \qquad (80)$$

which again reduces to

$$P[AB] = -\frac{1}{8\pi^3 i}\int_L\frac{\phi_B(v)}{v}\left(\int_L\frac{\phi_A(w-v)}{w-v}\left[\int_L\frac{\Psi_A(-u)\Psi_B(u-w)+\Psi_B(-u)\Psi_A(u-w)}{u-w}\,du\right]dw\right)dv \qquad (81)$$

and the indentations are the same as for Equation (78). Again, Thompson provides an example using all negative exponential density functions.[20] When these assumed functions are inserted in Equations (78) and (81) and are integrated, the results check with his.

c. In the next duel, we have the more difficult case where a duelist withdraws when his weapon fails but only after he discovers it, which is on the next attempted firing. For this case,[21] Thompson provides

$$P[A] = \int_0^{\infty} h_A(t)\left(\int_t^{\infty} r_A(\xi)d\xi\right)\left[\int_0^t\left(\int_x^{\infty} r_B(y)dy\right)\left(\int_{t-x}^{\infty} f_B(y)dy\right)\left(\sum_{n=1}^{\infty} q_B^n f_B^{n*}(x)\right)dx + \int_t^{\infty} f_B(x)dx\right]dt \qquad (82)$$

We note here that

$$\sum_{n=1}^{\infty} q_B^n f_B^{n*}(x)\,dx = \frac{q_B}{p_B}\,h_B(x)$$

and shall hence forward use that fact.

---

[20] *Ibid, p586*

[21] *Ibid, p588, Equations (6), (7), and (8)*

Rewriting Equation (82), we obtain

$$P[A] = \frac{q_B}{p_B} \int_0^{\infty} h_A(t) \left(\int_t^{\infty} r_A(\xi) d\xi \right) \left[\int_0^t \left(\int_x^{\infty} r_B(y) dy \right) \left(\int_{t-x}^{\infty} f_B(y) dy \right) h_B(x) dx \right] dt$$

$$+ \int_0^{\infty} h_A(t) \left(\int_t^{\infty} r_A(\xi) d\xi \right) \left(\int_t^{\infty} f_B(x) dx \right) dt . \quad (83)$$

We shall now do this one step at a time. First, from Equation (55), the last expression in Equation (83) becomes

$$\int_0^{\infty} h_A(t) \left(\int_t^{\infty} r_A(\xi) d\xi \right) \left(\int_t^{\infty} f_B(x) dx \right) dt$$

$$= \frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \left(\frac{\phi_B(-u) - 1}{u} \right) \left(\int_{-\infty}^{+\infty} \frac{\phi_A(u-w)[\Psi_A(w) - 1]}{w} dw \right) du . \quad (84)$$

Now, let us use Equation (61) on the inner part of the first expression in Equation (83).

$$\int_0^t \left(\int_x^{\infty} r_B(y) dy \right) \left(\int_{t-x}^{\infty} f_B(y) dy \right) h_B(x) dx =$$

$$\frac{1}{4\pi^2} \int_{-\infty}^{+\infty} \frac{e^{-ivt} - 1}{v} \left[\int_{-\infty}^{+\infty} \frac{e^{-itu} [\phi_B(u) - 1]}{u} \left(\int_{-\infty}^{+\infty} e^{i(v+u)x} h_B(x) \left\{ \int_x^{\infty} r_B(y) dy \right\} dx \right) du \right] dv . \quad (85)$$

We now use Equation (63) on the integration with respect to x in Equation (85) to get

$$\text{LHS} = \frac{1}{8\pi^3 i} \int_{-\infty}^{+\infty} \frac{(e^{-ivt} - 1)}{v} \left[\int_{-\infty}^{+\infty} \frac{e^{-itu}[\phi_B(u)-1]}{u} \left(\int_{-\infty}^{+\infty} \frac{\phi_B(v+u-\rho)[\Psi_B(\rho)-1]}{\rho} d\rho \right) du \right] dv . \quad (86)$$

Now, by writing out the entire first term of Equation (83) and reversing the order of integration, we get

LHS of the first term of Equation (83) =

$$\frac{q_B}{p_B\,8\pi^3 i}\int_{-\infty}^{+\infty}\frac{1}{v}\left\{\int_{-\infty}^{+\infty}\left(\frac{\phi_B(u)-1}{u}\right)\left(\int_{-\infty}^{+\infty}\frac{\phi_B(v+u-\rho)[\Psi_B(\rho)-1]}{\rho}d\rho\right)\left(\int_0^{\infty}(e^{-i(v+u)t}\;\;e^{-iut})h_A(t)\right.\right.$$

$$\left.\left.\bullet\left[\int_t^{\infty}r_A(\xi)d\xi\right]dt\right)du\right\}dv. \quad (87)$$

The inner integral of Equation (87) is from the application of Equations (42) and (53), therefore

$$\int_0^{\infty}(e^{-i(v+u)t}-e^{-iut})h_A(t)\left(\int_t^{\infty}r_A(\xi)d\xi\right)dt$$

$$=\frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{[\Psi_A(w)-1]}{w}\left\{\phi_A(-v-u-w)-\phi_A(-u-w)\right\}dw \quad (88)$$

Now, substitute Equation (88) into Equation (87) and then Equations (87) and (84) into Equation (83), thus

$$P[A]=-\frac{q_B}{p_B\;16\pi^4}\int_{-\infty}^{+\infty}\frac{1}{v}\left\{\int_{-\infty}^{+\infty}\frac{[\phi_B(u)-1]}{u}\left(\int_{-\infty}^{+\infty}\frac{\phi_B(v+u-\rho)[\Psi_B(\rho)-1]}{\rho}d\rho\right)\bullet\right.$$

$$\left.\left(\frac{[\Psi_A(w)-1]}{w}\left[\phi_A(-v-u-w)-\phi_A(-u-w)\right]dw\right)du\right\}dv$$

$$+\frac{1}{4\pi^2}\int_{-\infty}^{+\infty}\frac{[\phi_B(-u)-1]}{u}\left(\int_{-\infty}^{+\infty}\frac{\phi_A(u-w)[\Psi_A(w)-1]}{w}dw\right)du\,. \quad (89)$$

Equation (89) is now further reduced using the properties of positive random variables to

$$P[A] = -\frac{q_B}{p_B\, 16\pi^4}\int_U \frac{1}{v}\left\{\int_U \frac{[\phi_B(u)-1]}{u}\left(\int_L \frac{\phi_B(v+u-\rho)\gamma_B(\rho)}{\rho}\,d\rho\right)\left(\int_L \frac{\phi_A(-v-u-\omega)\gamma_A(\omega)}{\omega}\,d\omega\right)du\right\}dv$$

$$+\frac{1}{4\pi^2}\int_U \frac{\phi_B(-u)}{u}\left(\int_L \frac{\phi_A(u-\omega)\gamma_A(\omega)}{\omega}\,d\omega\right)du\,. \qquad (90)$$

This expression again checks Thompson's example[22] using all negative exponential pdf's. The prior results were based on a failure mechanism that was a random function of time. In what follows, the failures only occur at the instants of weapon firings.

d. The no-withdrawal case[23] is identical to the author's random ammunition-limited duel, and the characteristic function form of the solutions is given in the referenced paper and will not be repeated here.

e. Similarly, in the same paper mentioned above, the characteristic function form of the solution to the duel in which a contestant withdraws the instant a failure occurs is also given and will not be pursued further.

f. The case in which a contestant whose weapon has failed on the n^{th} round but who only discovers that fact on the (n+1)st round and then withdraws is provided by Thompson[24] as

$$P[A] = \int_0^\infty h_A(t)\left(\int_t^\infty f_B(x)dx + \sum_{n=1}^{\infty} q_B^n \int_0^t f_B^{n*}(x)\left[\int_{t-x}^\infty f_B(y)dy\right]dx\right)dt\,. \quad (91)$$

Thompson assumes the probability of failure on any given round is fixed at u and, therefore, $p + q + u = 1$. Again, noting that

$$\sum_{n=1}^{\infty} q_B^n f_B^{n*}(x)\,dx = \frac{q_B}{p_B}\, h_B(x)$$

---

[22] *Ibid*, p589

[23] *Ibid*, p594

[24] *Ibid*, pp 597, 598

we rewrite Equation (91) as

$$P[A] = \int_0^\infty h_A(t)\left(\int_t^\infty f_B(x)dx\right) dt + \frac{q_B}{p_B}\int_0^\infty h_A(t)\left[\int_0^t h_B(x)\left(\int_{t-x}^\infty f_B(y)dy\right) dx\right]dt . \quad (92)$$

Using Equation (54) in the first integral and Equation (61) in the second, we have

$$P[A] = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\phi_A(-u)[\phi_B(u)-1]}{u} du +$$

$$\frac{q_B}{p_B\, 4\pi^2}\int_0^\infty h_A(t)\left(\int_{-\infty}^{+\infty}\frac{[e^{-iut}-1]}{u}\left[\int_{-\infty}^{+\infty}\frac{\phi_B(u+w)\; e^{-iwt}[\phi_B(w)-1]}{w} dw\right]du\right)dt . \quad (93)$$

In the second integral, we now reverse the order of integration and integrate out t to get

$$P[A] = \frac{1}{2\pi i}\int_{-\infty}^{+\infty}\frac{\phi_A(-u)[\phi_B(u) - 1]}{u} du$$

$$+ \frac{q_B}{p_B 4\pi^2}\int_{-\infty}^{+\infty}\frac{1}{u}\left[\int_{-\infty}^{+\infty}\frac{\phi_B(u+w)[\phi_B(w) - 1]}{w}\left(\phi_A(-u-w) - \phi_A(-w)\right)dw\right]du . \quad (94)$$

Now, taking advantage of the properties of positive RVs, Equation (94) becomes

$$P[A] = \frac{1}{2\pi i}\int_L \phi_A(-u)\phi_B(u)du/u$$

$$+ \frac{q_B}{p_B 4\pi^2}\int_L\frac{[\phi_B(w)-1]}{w}\left(\int_U\frac{\phi_B(u+w)\phi_A(-u-w)}{u} du\right)dw . \quad (95)$$

The expression for a draw[25] is composed of two terms, each of which is similar to P[A] and the development is almost identical; therefore, we shall merely write down the results, i.e.,

$$P[AB] = \frac{u_A}{p_A 2\pi i}\int_L \phi_A(-u)\phi_B(u)\,du/u + \frac{u_B}{p_B 2\pi i}\int_L \phi_B(-u)\phi_A(u)\,du/u$$

$$+ \frac{u_A q_B}{p_A p_B 4\pi^2}\int_L \frac{[\phi_B(\omega)-1]}{\omega}\left(\int_U \frac{\phi_B(u+\omega)\phi_A(-u-\omega)}{u}\,du\right)d\omega$$

$$+ \frac{u_B q_A}{p_B p_A 4\pi^2}\int_L \frac{[\phi_A(\omega)-1]}{\omega}\left(\int_U \frac{\phi_A(u+\omega)\phi_B(-u-\omega)}{u}\,du\right)d\omega\,. \qquad (96)$$

Equations (95) and (96) both check with one of Thompson's examples[26] where again he uses negative exponential pdf's.

g. The remainder of Thompson's paper deals exclusively with negative exponential pdf's from the beginning, and all the results are consequently so specialized to that function that no further simplifications through the use of characteristic functions are possible.

h. In a paper published in the Naval Research Logistics Quarterly, Thompson extends the situation in paragraph f above to the case where each contestant has an independent probability distribution on the possibility of a failure on the nth round (and thus a withdrawal on the n+1st round).[27] He provides the following as solutions:

If α_j = P [A's weapon fails on round j+1]

β_k = P [B's weapon fails on round k+1] (97)

and $\sum_{j=0}^{\infty} \alpha_j = \sum_{k=0}^{\infty} \beta_k = 1$

[25] *Ibid, p599*

[26] *Ibid, pp599, 600*

[27] *David E. Thompson, Stochastic Duels Involving Reliability, The Naval Research Logistics Quarterly, Vol 19, No. 1, March 1972, pp 147-148, Equations (6) through (11).*

then

$$P[A] = \int_0^\infty \left\{ \sum_{n=1}^\infty p_A q_A^{n-1} f_A^{n*}(t) \left[\sum_{j=n}^\infty \alpha_j \right] \right\} \left\{ \int_t^\infty f_B(x)dx + \right.$$

$$\left. \sum_{n=1}^\infty \left[\sum_{k=n}^\infty \beta_k \right] q_B^n \int_0^t f_B^{n*}(x) \left[\int_{t-x}^\infty f_B(y)dy \right] dx \right\} dt \qquad (98)$$

and

$$P[AB] = \int_0^\infty \left\{ \int_t^\infty f_B(x)dx + \sum_{n=1}^\infty \left[\sum_{k=n}^\infty \beta_k \right] q_B^n \int_0^t f_B^{n*}(x) \left[\int_{t-x}^\infty f_B(y)dy \right] dx \right\} \cdot$$

$$\left\{ \sum_{j=0}^\infty \alpha_j \, q_A^{j+1} \, f_A^{(j+1)*}(t) \right\} dt$$

$$+ \int_0^\infty \left\{ \int_t^\infty f_A(x)dx + \sum_{n=1}^\infty \left(\sum_{j=n}^\infty \alpha_j \right) q_A^n \int_0^t f_A^{n*}(x) \left[\int_{t-x}^\infty f_A(y)dy \right] dx \right\} \cdot$$

$$\left\{ \sum_{k=0}^\infty \beta_k \, q_B^{k+1} \, f_B^{(k+1)*}(t) \right\} dt \qquad (99)$$

Now, if we define

$$h_A(t) = \sum_{n=1}^\infty p_A q_A^{n-1} f_A^{n*}(t) \qquad (100)$$

and

$$h_A'(t) = \sum_{n=1}^\infty p_A q_A^{n-1} f_A^{n*}(t) \left(\sum_{j=n}^\infty \alpha_j \right) \qquad (101)$$

we then have the corresponding cf's to be

$$\Phi_A(u) = \frac{p_A \phi_A(u)}{1 - q_A \phi_A(u)} \tag{102}$$

and reducing the double sum in Equation (101) to a single sum

$$\Phi'_A(u) = \Phi_A(u)\left[1 - \sum_{j=0}^{\infty} \alpha_j \, q_A^j \, \phi_A^j(u)\right] . \tag{103}$$

If we now define N_A to be the random variable, the round number on which A's weapon fails, then the probability generating function (z-transform) of N_A is

$$P_{N_A}(z) = \sum_{j=0}^{\infty} \alpha_j z^{j+1} \tag{104}$$

and thus

$$\Phi'_A(u) = \Phi_A(u)\left[1 - \frac{P_{N_A}[q_A \phi_A(u)]}{q_A \phi_A(u)}\right] \tag{105}$$

Similar expressions can be written for B. Then, using Equations (54), (61), (104), and (105) and integrating out t first, we get

$$P[A] = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \Phi_A(-u) \left[1 - \frac{P_{N_A}[q_A \phi_A(-u)]}{q_A \phi_A(-u)}\right] \left[\Phi_B(u) - 1\right] \frac{du}{u}$$

$$+ \frac{q_B}{4\pi^2 p_B} \int_{-\infty}^{+\infty} \frac{1}{u} \left\{ \left(\int_{-\infty}^{+\infty} \Phi'_B(u+\omega)[\Phi_B(\omega) - 1] \right) \left(\Phi'_A(-u-\omega) - \Phi'_A(-\omega) \right) \frac{d\omega}{\omega} \right\} du . \tag{106}$$

This may now be reduced to final form by integrating u first using Equation (105) and the properties of our characteristic functions

$$P[A] = \frac{1}{2\pi i}\int_{L}\phi_A(-u)\left[1 - \frac{P_{N_A}[q_A\phi_A(-u)]}{q_A\phi_A(-u)}\right]\phi_B(u)\,\frac{du}{u}$$

$$+ \frac{q_B}{4\pi^2 p_B}\int_{L}\frac{[\phi_B(w) - 1]}{w}\left\{\int_{U}\phi_B(u+w)\left[1 - \frac{P_{N_B}[q_B\phi_B(u+w)]}{q_B\phi_B(u+w)}\right]\bullet\right.$$

$$\left.\phi_A(-u-w)\left[1 - \frac{P_{N_A}[q_A\phi_A(-u-w)]}{q_A\phi_A(-u-w)}\right]\frac{du}{u}\right\}dw\ . \qquad (107)$$

Proceeding in the same fashion, we obtain the P[AB] as shown below.

$$P[AB] = \frac{1}{2\pi i}\int_{L}P_{N_A}[q_A\phi_A(-u)]\ \phi_B(u)\ \frac{du}{u}\ +$$

$$+ \frac{q_B}{4\pi^2 p_B}\int_{L}\frac{[\phi_B(w) - 1]}{w}\left\{\int_{U}\phi_B(u+w)\left[1 - \frac{P_{N_B}[q_B\phi_B(u+w)]}{q_B\phi_B(u+w)}\right]P_{N_A}[q_A\phi_A(-u-w)]\ \frac{du}{u}\right\}dw$$

$$+ \frac{1}{2\pi i}\int_{L}P_{N_B}[q_B\phi_B(-u)]\ \phi_A(u)\ \frac{du}{u}$$

$$+ \frac{q_A}{4\pi^2 p_A}\int_{L}\frac{[\phi_A(w) - 1]}{w}\left\{\int_{U}\phi_A(u+w)\left[1 - \frac{P_{N_A}[q_A\phi_A(u+w)]}{q_A\phi_A(u+w)}\right]P_{N_B}[q_B\phi_B(-u-w)]\ \frac{du}{u}\right\}dw\ . \qquad (108)$$